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13. ABSTRACT

The frictional identation problem for an elastic half space

is formulated in terms of a coupled pair of singular integral equations,

leading to determination of adhesive boundary as an eigenvalue of Fredholm

equation with positive kernel.

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AN EIGENVALUE PROBLEM FOR ELASTIC
CONTACT WITH FINITE FRICTION

D. A. Spence

# ABSTRACT

The two-dimensional indentation of an elastic half space by a
rigid punch under a slowly-applied normal load is considered, for
the case in which there is a finite coefficient of friction $\mu$ between
the surfaces. The contact area is then divided into an inner ad-
hesive region $-c < x < c$ in which the surface displacements are
known, surrounded by regions $c < |x| < 1$ in which the friction is
limiting and the lateral displacement (which must increase in
proportion to the overall load) is not known in advance. The
problem is formulated in terms of a coupled pair of singular integral
equations for the normal and shear stresses $\sigma$ and $\tau$ at the surface;
these are combined to give a single homogeneous Fredholm equation
with positive kernel for a quantity $\phi$ proportional to the difference
$\tau - \mu\sigma$ in the adhesive region. The largest eigenvalue of this
equation, for which $\phi > 0$, gives the adhesive boundary $c$ in terms
of $\mu$ and Poisson's ration $\nu$. A similarity transformation shows that
$c$ has the same value for both flat-faced and power law punches.

# AN EIGENVALUE PROBLEM FOR ELASTIC CONTACT WITH FINITE FRICTION

## D. A. Spence[†]

## 1. Introduction

The problem posed by the indentation of an elastic half space by a rigid punch has a straightforward solution in the case of frictionless contact, for which a boundary condition of zero shear stress beneath the punch applies. As a step towards the physical situation in which the shear stress is controlled by friction, solutions have also been found in a number of cases for the opposite limit of fully adhesive contact, in which no relative slip between the indentor and the half-space is allowed once contact has been established. The simplest such case is the symmetrical two-dimensional indentation of a half space by a flat punch over the interval $-1 < x < 1$ considered by Muskhelishvili (1953 pp. 475-7). The solution of the linear elastic equations in this case, although physically realistic near the centre of the punch, is not so near the edges, where the ratio of shear stress $\tau$ to normal stress $\sigma$ at the surface, given by $\tan\left[\frac{1}{2}\kappa\log\left(\frac{1+|x|}{1-|x|}\right)\right]$ where $\kappa$ is a material constant, is divergent and oscillatory. This behaviour would be modified either by slip or, if this were prevented, by plastic deformation near the edges.

In the present paper, the same problem is treated under the more general assumption of finite friction, with coefficient $\mu$, between the surfaces. The

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case of the flat punch is indicated in figure 1a (p. 33). We suppose that over a central part C'C of the contact area, the friction is sufficient to prevent slip taking place as the load P is slowly increased, but that in the remaining parts A'C' and CA, the friction is limiting so that the material is displaced inwards as well as downwards under progressive application of the load. It is assumed that

(i) the load is applied sufficiently slowly for static equilibrium to apply at each stage;

(ii) it is applied monotonically, so that slip is always in the same direction.

At the outset, it is not clear that a solution of the simple form indicated will exist, since there might have been more than one region of adhesion, with interspersed regions of slip (and this may well be the configuration to be looked for in the corresponding "unloading" problem). But we are able to show that the problem as posed possesses a unique solution satisfying the linear elastic equations and the physical boundary conditions. The first of these, over the interval C'C, is imposed in the form of an inequality on the stresses

$$|\tau| \leq \mu |\sigma| \qquad (1.1)$$

together with a requirement of zero displacement parallel to the interface, while on the intervals A'C' and CA we set

$$|\tau| = \mu |\sigma| \qquad (1.2)$$

but cannot specify the parallel displacement u, which must finally be found as part of the solution.

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$\mu$ is then effectively the limiting value of the coefficient of sliding friction as the relative speed goes to zero. This may possibly be lower than the coefficient of static friction, but if the inequality (1.1) is satisfied when $\mu$ has its sliding value, it necessarily holds also if the static coefficient is higher.

The extent of the slip region, given by the ratio $CC'/AA' = c$ say is an eigenvalue of the problem which cannot be specified in advance, but we find that for given values of the physical constants ($\mu$ and the Poisson ratio $\nu$) there is just one value of $c$ for which the inequality can be satisfied. $c$ is identified as an eigenvalue of a certain homogeneous Fredholm equation with a positive kernel, and the general theory of positive linear forms shows that only one eigenfunction of such an equation is everywhere non-negative, so $c$ can be uniquely identified as the corresponding eigenvalue.

Our main object in the present paper is to formulate the mathematical problem leading to this eigenvalue, and to determine its value precisely. This provides an analytical check on the accuracy of numerical methods which are potentially able to give detailed information about the distributions of stress under the punch more easily. For this reason the stress distributions are not explicitly calculated here except in limiting cases. A numerical approach to the problem that will be reported separately is indicated in section 5.3.

The governing equations are set out in section 2. Before proceeding to the solution for the flat punch, we note also the form taken by the equations for the more general case of a power-law indentor $z = Br^n$ as indicated

in figure 1b (e.g. a circular cylinder (n = 2) or a flat wedge (n = 1), and show by appeal to a similarity argument given in an earlier paper (Spence 1968a) that the solution for the stresses in this case can be obtained by quadrature from that for the flat punch. It follows that the extent of the slip region is the same in both cases, despite the different forms of the distribution of normal stress. Thus the present solution of the eigenvalue problem covers the whole range of self-similar indentors with symmetrical loading.

A useful limiting approximation to the slip problem can be obtained by neglecting the change in the normal stress from its frictionless distribution due to the presence of shear forces. This is calculated in section 2.2 and leads to an expression for c similar in form to that given by Galin (1945) in an approximate but much earlier treatment of the present problem. In section 3, the full linear equations are treated; these are reduced to a single Fredholm equation for a quantity $r(x)$ proportional to the difference $\mu|\sigma| - (\tau)$ within the adhesive region, treating its extent as known. A change of variable relates c to an eigenvalue of the Fredholm equation, as shown in section 4, and the results of calculations of the numerical value of c as a function of $\mu$ and $\nu$ are outlined in section 5.

## 2. Governing equations

Suppose $ax$, $ay$ are physical coordinates parallel and normal to the surface of an elastic half-space, the surface values of the normal and shear stresses being denoted by

$$(\sigma_{yy})_{y=0} = -(P/a)\, p(x), \quad (\tau_{xy})_{y=0} = (P/a)q(x)$$

where P is the total applied normal force over the contact area $|x| < 1$, so that

$$\int_{-1}^{1} p(x)\,dx = 1 \qquad (2.1)$$

and let $au(x)$, $av(x)$ be the corresponding surface displacements due to the load. Then it follows from results of Muskhelishvili (1953, chapter 19) that p, q, u, v are connected by the coupled pair of singular integral equations

$$\frac{1}{\pi} \int_{-1}^{1} \frac{p(t)dt}{t-x} + \gamma q(x) = -(\frac{G}{1-\nu})\frac{a}{p}\frac{dv}{dx} \qquad (2.2)$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{q(t)dt}{t-x} - \gamma p(x) = (\frac{G}{1-\nu})\frac{a}{p}\frac{du}{dx} \qquad (2.3)$$

in which $\gamma = (1-2\nu)/(2-2\nu)$, $\nu$ being Poisson's ratio and G the shear modulus. If the load is applied symmetrically about $x = 0$, p and u' are even functions of x, q and v' are odd.

The boundary conditions on the normal displacements are

$$-1 < x < 1: \ (\frac{G}{1-\nu})\frac{a}{p}\frac{dv}{dx} = \begin{cases} 0 & \text{(flat punch, figure 1a)} \qquad (2.4) \\ A|x|^{n-1}(\text{sgn}\,x) & \text{(power law indentor } z = Br^{n}, \\ & \text{figure 1b)}, \end{cases}$$

where $A = (\frac{G}{1-\nu})(\frac{a}{p})^{n}\,nB$ is a constant, initially unknown, relating the geometry of the indentor and the size of the contact region to the normal force.

The lateral displacement within the adhesive region is zero for the flat punch; and for the power law indentor similarity considerations of the type given for the fully adhesive indentor by Spence (1968a) continue to apply in

the case of partial slip,[†] giving the second boundary condition

$$-c < x < c: \quad \left(\frac{G}{1-\nu}\right)\frac{a}{p}\frac{du}{dx} = \begin{cases} 0 & \text{(flat punch)} \\ C|x|^{n-1} & \text{(power law)} . \end{cases} \quad (2.5)$$

In the remaining intervals $c < |x| < 1$, $\dfrac{du}{dx}$ can not be specified. The values of $A$ and $C$ are eventually to be determined from the conditions that the stresses vanish at the boundaries $x = \pm 1$ of the contact region. (For the flat punch, by contrast, the stresses become infinite at these edges).

The boundary conditions on the stresses are neatly expressible in terms of the difference

$$r(x) = p(x) - \frac{q(x)}{\mu}(\text{sgn } x) . \quad (2.6)$$

To prevent slip in the adhesive region we require

$$r(x) \geq 0 \qquad (-c < x < c) \quad (2.7a)$$

while in the regions of slip

$$r(x) = 0 \qquad (c < |x| < 1) . \quad (2.7b)$$

In these boundary conditions, $c$ is an eigenvalue to be determined in terms of the physical constants $\mu, \gamma$ .

## 2.1. Reduction of "power law" equations to the flat punch problem

Before investigating the eigenvalue problem, we note that the equations for the power law indentor are reducible to those for the flat punch. Using the symmetry of $p$ and antisymmetry of $q$ to write them on the interval $(0,1)$,

[†] This point will be discussed further in a subsequent report.

they become for $x > 0$

$$0 < x < 1: \quad \frac{2x}{\pi} \int_{-1}^{1} \frac{p(t)dt}{t^2 - x^2} + \gamma q(x) = -Ax^{n-1} \tag{2.8}$$

$$0 < x < c: \quad \frac{2}{\pi} \int_{0}^{1} \frac{tq(t)dt}{t^2 - x^2} - \gamma p(x) = Cx^{n-1} \tag{2.9}$$

and the condition that the stresses vanish at the edge is

$$p(1) = 0 = q(1) . \tag{2.10}$$

Application of the differential operator $\frac{n-1}{n} - \frac{x}{n} \frac{d}{dx}$ to these equations removes the constants A and C, reducing them to

$$0 < x < 1: \quad \frac{2x}{\pi} \int_{0}^{1} \frac{p_0(t)dt}{t^2 - x^2} + [\gamma q_0(x)] = 0 \tag{2.11}$$

$$0 < x < c: \quad \frac{2}{\pi} \int_{0}^{1} \frac{t q_0(t)dt}{t^2 - x^2} - \gamma p_0(x) = 0 \tag{2.12}$$

respectively, where

$$p_0(x) = p(x) - \frac{1}{n}(xp)', \quad q_0(x) = q - \frac{1}{n}(xq)' . \tag{2.13}$$

These are exactly the equations that apply in the flat punch case, so the solution for the power law indentor is given by quadrature, using the boundary condition (2.10), as

$$p(x) = nx^{n-1} \int_{x}^{1} \frac{p_0(t)dt}{t^n}, \quad q(x) = nx^{n-1} \int_{x}^{1} \frac{q_0(t)dt}{t^n} . \tag{2.14}$$

If the distribution of $p_0(x)$ is normalised in accordance with (2.1), that of $p(x)$ is automatically so, since integration of (2.13) gives

$$\int_0^1 p(x)dx = \int_0^1 p_0(x)dx \ .$$

Moreover, the distribution of $p$ and $q$ so found satisfy the frictional boundary conditions (2.7a and b) with the same value of $c$ as for the flat punch, since

$$r(x) = nx^{n-1} \int_x^1 t^{-n} \left[ p_0(t) - \frac{q_0(t)}{\mu} \right] dt \begin{cases} = 0 & c < x < 1 \\ \geq 0 & 0 < x < c \ . \end{cases}$$

Therefore the solution of the eigenvalue problem for the flat punch also gives the solution for a power law indentor, and we shall from now on only treat the former case, and work from equations (2.11), and (2.12).

## 2.2. A limiting solution

Since $|q/p|$ is at most equal to $\mu$, the term in square brackets in equation (2.11) is of order $\mu \gamma$ times the normal stress term, and a first approximation that throws some light on the physics can be found by omitting the term, when (2.11) has the normalised solution

$$p_0(x) = (1/\pi)(1 - x^2)^{-\frac{1}{2}} \ . \tag{2.15}$$

This approximation, equivalent to neglecting the effect of shear forces on the distribution of normal pressure, has been used for the calculation of the shear forces themselves in other contexts (e.g. Goodman 1962, Mindlin 1949). On substitution of (2.15), (2.12) becomes a singular integral equation for $q_0(x)$ on the interval $0 < x < c$, since we can write $q_0 = \mu p_0$ in $(c, 1)$:

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$$\frac{2}{\pi} \int_0^c \frac{t\, q_0(t)dt}{t^2 - x^2} + \chi(x) = 0 \qquad (2.16)$$

where

$$\chi(x) = \frac{2\mu}{\pi} \int_0^1 \frac{t\, p_0(t)dt}{t^2 - x^2} - \gamma p_0(x) \ .$$

Inversion on the interval $(0, c)$ gives the solution bounded at $x = 0$ as

$$q_0(x) = \frac{2x}{\pi} \int_0^c \left( \frac{c^2 - y^2}{c^2 - x^2} \right)^{\frac{1}{2}} \frac{\chi(y)dy}{y^2 - x^2} \ . \qquad (2.17)$$

The solution is in general unbounded at $x = c$, but is bounded if $\int_0^c (c^2 - y^2)^{-\frac{1}{2}} \chi(y)dy = 0$, and since one of the terms of $\chi$ is proportional to $\mu$ and the other to $\gamma$, the vanishing of this integral gives an expression for the ratio $\mu/\gamma$:

$$\mu/\gamma = K(c)/K'(c), \qquad (2.18)$$

where $K(c)$ is the complete elliptic integral $\int_0^{\pi/2} (1 - c^2 \sin^2\theta)^{-\frac{1}{2}} d\theta$, and $K'(c) = K(\sqrt{1 - c^2})$.

Galin's result, obtained by a lengthy analysis using approximations in a complex plane that are difficult to interpret physically, is of the same form. In his equation 12.36, the right hand side is identical with the above, but the left, in our notation, is $(\tan^{-1}\mu)/\frac{1}{2} \log\left(\frac{1+\gamma}{1-\gamma}\right)$, which is the same when $\mu$ and $\gamma$ are both small compared with $1$. The accuracy of the two solutions relative to the exact solution is indicated in figure 2. Galin's is very close for $\nu = 0.3$, i.e. $\gamma = 2/7$, and reasonably so for $\nu = 0$, $\gamma = \frac{1}{2}$; equation (2.18) is not as close as Galin's but is still useful numerically, and knowledge of

the limit as $\mu \to 0$, $\gamma \to 0$ with $\mu/\gamma$ fixed is also useful in the subsequent analysis.

Asymptotic expressions when $c$ is small, and when $c$ is close to $1$, are found from the limiting behaviour of the elliptic integrals:

$$K(c)/K'(c) \sim \begin{cases} (2/\pi) \log(4/\sqrt{1-c^2}) & \text{as } c \to 1 \\ (\pi/2) \log(4/c) & \text{as } c \to 0, \end{cases}$$

so that the width of the adhesive region is given, according to (2.18), by

$$C = \begin{cases} 4e^{-\pi\gamma/2\mu} & \text{as } \mu \to 0 \quad \text{for fixed } \gamma \\ 1 - 8e^{-\pi\mu/\gamma} & \text{as } \mu \to \infty \quad \text{for fixed } \gamma. \end{cases} \tag{2.19}$$

When (2.18) is satisfied, a calculation using elliptic functions (appendix D) shows that

$$q_0(x) = [\mu/\pi K(c)] (1 - x^2)^{-\frac{1}{2}} F(\sin^{-1} \frac{x}{c}, c) \tag{2.20}$$

where $F(\xi, c)$ is the incomplete elliptic integral $\int_0^\xi (1 - c^2 \sin^2\theta)^{-\frac{1}{2}} d\theta$. In the further limit $c \to 0$, $F(\xi) \to \xi$ and $K(c) \to \pi/2$, so

$$\frac{q_0(x)}{\mu p_0(x)} = \frac{2}{\pi} \sin^{-1} \frac{x}{c} \tag{2.21}$$

in the adhesive region $0 < x < c$. Figure 3 shows computed curves of $q/\mu p$ for the case $\nu = 0$ for three values of $c$, for comparison with this expression. The agreement is best for the smallest value of $c$. The computed curves for $\nu = 0.3$ would lie approximately mid way between those for $\nu = 0$ and the broken curves.

## 3. Reduction of the dual problem to a Fredholm equation

In this section we treat the flat punch equations (2.11), (2.12) but will from now on omit the suffix $0$. If $q$ is eliminated in favour of $r$, these become

$$0 < x < 1: \quad p(x) + 2x/\pi)(\cot \pi a)\int_0^1 \frac{p(y)dy}{y^2 - x^2} = r(x) \tag{3.1}$$

$$0 < x < c: \quad -\gamma^2 p(x) + (2/\pi)(\tan \pi a)\int_0^1 \frac{p(y) - r(y)}{y^2 - x^2}\, y dy = 0, \tag{3.2}$$

where
$$\tan \pi a = \mu \gamma \qquad (0 < a < \tfrac{1}{2}),$$

which are to be solved subject to the requirement that $r$ vanishes on $(c, 1)$ and is bounded and non-negative on $(0, c)$ in accordance with (2.7). A singularity in $p$ at $x = 1$ can however be admitted in the solution.

In the physical problem it would be natural to fix the values of $a$ and $\gamma$, and try to find $c$. However, it is more convenient analytically to proceed for the present as if all three were known, but we shall find in due course that a solution to (3.1), (3.2) satisfying the further requirement that $r(x)$ is bounded and non-negative on $(0, c)$ exists only when a certain combination of $a$, $\gamma$ and $c$ is equal to the first eigenvalue of the Fredholm equation (4.7) below. (3.1) is a singular integral equation for $p(x)$ in terms of $r(x)$, with solution obtainable by standard methods (appendix A) as

$$0 < x < 1: \quad p(x) = (\sin \pi a)^2 r(x) - x\left(\frac{\sin 2\pi a}{\pi}\right)\int_0^c \frac{\omega(y)}{\omega(x)} \frac{r(y)dy}{\left(y^2 - x^2\right)} \tag{3.3}$$

where $\omega(x) = \left(\frac{1-x^2}{x^2}\right)^{\frac{1}{2}-a}$.

(An arbitrary multiple of $1/x\omega(x)$ forms part of the solution, but must be excluded since it is unbounded at $x = 0$. The remaining integral is regular at $x = 0$, but behaves like $(1 - x^2)^{-\frac{1}{2}+\alpha}$ near $x = 1$.)

This expression for $p(x)$ can now be substituted into (3.2) to yield a homogeneous equation for $r$ alone, as follows:

(i) On substitution of $p$ into the second term of (3.2), the equation becomes

$$-\gamma^2 p(x) - \left(\frac{\sin 2\pi\alpha}{\pi}\right) \int_0^c \frac{y r(y) dy}{y^2 - x^2} = 4\left(\frac{\sin\pi\alpha}{\pi}\right)^2 \int_0^1 \frac{y^2 dy}{y^2 - x^2} \int_0^c \frac{\omega(u)}{\omega(y)} \frac{r(u) du}{u^2 - y^2} . \qquad (3.4)$$

(ii) The order of integration on the right hand side may be reversed by use of the Bertrand-Poincaré lemma (Tricomi 1957 p. 172) when the term becomes

$$(\sin\pi\alpha)^2 \left[ -r(x) + \left(\frac{2}{\pi}\right)^2 \int_0^c \omega(u) \, r(u) du \int_0^1 \frac{y^2 dy}{\omega(y)(y^2 - x^2)(u^2 - y^2)} \right].$$

The inner integral here can be split into partial fractions and evaluated by use of a general result derived in Appendix B :

$$2x^2 \left(\frac{\sin\pi\alpha}{\pi}\right) \int_0^1 \frac{dy}{\omega(y)(y^2 - x^2)} = \frac{x \cos\pi\alpha}{\omega(x)} - x^2 g(\alpha) A(x^2) , \qquad (3.5)$$

where $A(x^2) = F(1, \frac{1}{2}; 1+\alpha; x^2)$ and $g(\alpha) = \Gamma(\frac{1}{2} + \alpha)/\Gamma(1 + \alpha) \Gamma(\frac{1}{2})$ .

(iii) When these results are combined, (3.4) reduces to

$$-(\sin\pi\alpha)^2 r(x) - \frac{\sin 2\pi\alpha}{\pi} \int_0^c \left\{ x - \frac{u\omega(u)}{\omega(x)} \right\} \frac{r(u) du}{u^2 - x^2} + \frac{2}{\pi} g(\alpha) \sin\pi\alpha \int_0^c G(x^2, u^2) \omega(u) r(u) du,$$

where

$$G(x^2, u^2) = [x^2 A(x^2) - u^2 A(u^2)]/(x^2 - u^2) = \sum_{n=0}^{\infty} a_n (x^{2n} + x^{2n-2} u^2 + \ldots + u^{2n}), \qquad (3.6)$$

and $a_k = (\tfrac{1}{2})_k /(1+\alpha)_k$ is the coefficient of $z^k$ in the expansion of $A(z)$. The first term and the first part of the second term combine to produce exactly $-p(x)$, by (3.3), while the remainder of the second term cancels the second term on the left of (3.4), which therefore reduces to

$$0 < x < c: \quad (1 - \gamma^2)p(x) - \frac{2}{\pi} g(\alpha)(\sin \pi \alpha) \int_0^c G(x^2, u^2) \omega(u) r(u) du = 0 . \qquad (3.7)$$

Since $p(x)$ has already been expressed in terms of $r(x)$, (3.7) is equivalent to a Fredholm equation for $r$ on the interval $(0, c)$. It is reduced to canonical form in the next section.

In the slip region $c < x < 1$, the right hand side of (3.7) equals $(\frac{\gamma G}{-\nu})(\frac{a}{p})\frac{du}{dx}$, by (2.3); the displacement $u(x)$ could therefore be calculated once the equation has been solved for $r(x)$. It can be confirmed directly from (2.3) that $\frac{du}{dx}$ and therefore $u(x)$ are negative, i.e. that the displacements in the slipping regions are inwards. This is consistent with the positive sign of $q$ for $x > 0$ (shear stress away from the origin) since the frictional force opposes the motion.

## 4. The Fredholm equation for r(x)

In (3.3) and (3.7) write

$$x^2 = c^2\xi, \quad (1 - x^2)^{\frac{1}{2} - \alpha} r(x) = \phi(\xi), \quad (1 - x)^{\frac{1}{2} - \alpha} p(x) = \psi(\xi), \qquad (4.1)$$

Then the equations become

$$\psi(\xi) = (\sin \pi \alpha)^2 \phi(\xi) - \left(\frac{\cos \pi \alpha \sin \pi \alpha}{\pi}\right) \int_0^1 (\frac{\xi}{\eta})^{1-\alpha} \frac{\phi(\eta) d\eta}{\eta - \xi} \qquad (4.2)$$

$$\equiv L\phi \quad \text{say},$$

and

$$\psi(\xi) = \frac{g(a)\sin\pi a}{\pi(1-\gamma^2)} c^{2a} \int_0^1 \eta^{a-1} H(\xi,\eta)\phi(\eta)d\eta, \qquad (4.3)$$

in which  c  plays the role of an eigenvalue, to be fixed so that (4.2) and

(4.3) possess a solution such that

$$0 \le \phi(\xi) < \infty \qquad (4.4)$$

on (0, 1).  The kernel in (4.3) is

$$H(\xi,\eta) = (1-c^2\xi)^{\frac{1}{2}-a} G(c^2\xi, c^2\eta) = \sum_{m,n=0}^{\infty} c^{2(m+n)} h_{mn}\xi^m\eta^n \qquad (4.5)$$

say.

The inverse of (4.2) that is bounded on  (0, 1)  is

$$\phi(\xi) = \psi(\xi) + \frac{\cot\pi a}{\pi} \int_0^1 \left(\frac{\xi}{\eta}\right)^{\frac{1}{2}}\left(\frac{1-\xi}{1-\eta}\right)^{\frac{1}{2}-a} \frac{\psi(\eta)d\eta}{\eta-\xi} \equiv L^{-1}\psi \quad \text{say.} \qquad (4.6)$$

[This excludes an arbitrary multiple of  $\xi^{\frac{1}{2}}/(1-\xi)^{\frac{1}{2}-a}$,  to comply with (4.4)].

(4.3) and (4.6) give

$$\lambda\phi(\xi) = \int_0^1 \eta^{a-1} K(\xi,\eta)\phi(\eta)d\eta, \qquad (4.7)$$

where $K(\xi,\eta) = \left(\frac{\sin\pi a}{\pi g(a)}\right)L^{-1}(\xi) H(\xi,\eta) = \left(\frac{\sin\pi a}{\pi g(a)}\right) \sum c^{2(m+n)} h_{mn}\phi(\xi)\eta^n$ (4.8)

and $$\lambda = (1-\gamma^2)/c^{2a}[g(a)]^2 . \qquad (4.9)$$

Here $$\phi_m(\xi) = L^{-1}(\xi^m) = \xi^m\phi_0(\xi), \qquad (4.10)$$

$$+ g(a)(\cot\pi a)\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}-a} \sum_{r=0}^{m-1} a_{n-r-1}\xi^r$$

where $\phi_0(\xi) = L^{-1}(1)$, which is evaluated using the result in Appendix B, and can be written in two ways:

$$\left.\begin{array}{l} \phi_0(\xi) = [\Gamma(1-\alpha)/\Gamma(\tfrac{1}{2})\Gamma(\tfrac{3}{2}-\alpha)]\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}-\alpha} F(1, 1-\alpha; \tfrac{3}{2}-\alpha; 1-\xi) \\[2mm] \qquad = 1 - 2[\Gamma(1-\alpha)/\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}-\alpha)]\xi^{\frac{1}{2}} F(\tfrac{1}{2}, \tfrac{1}{2}+\alpha; \tfrac{3}{2}; \xi) \ . \end{array}\right\} \tag{4.11}$$

The first shows that $\phi_0$ and therefore all the $\phi_n$ tend to zero like $(1-\xi)^{\frac{1}{2}-\alpha}$ as $\xi \to 1$; the second shows that $\phi_0(0) = 1$, while $\phi_m(0) = 0$ for all $m > 0$. Insertion of these expressions in (4.8) shows that $K$ is bounded on $0 \le \xi, \eta \le 1$, and if we write $\phi(\xi) = \xi^{\frac{1}{2}(1-\alpha)}\phi^*(\xi)$ the equation is

$$\lambda\phi^*(\xi) = \int_0^1 (\xi\eta)^{-\frac{1}{2}(1-\alpha)} K(\xi, \eta)\phi^*(\eta)d\eta \tag{4.12}$$

in which, because of the boundedness of $K$, the kernel is square integrable provided $\alpha > 0$, so the standard Fredholm theory shows that solutions exist for at most a discrete set of eigenvalues $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ldots$ say. Moreover, all the coefficients $h_{mn}$ are $>0$ (this is confirmed by a detailed calculation in §5), and $\phi_m(\xi) > 0$ on the open interval, so $K$ is positive, and $\lambda_1$ is the Perron root, which is positive and is the maximum in absolute value; and the corresponding eigenfunction $\phi^{(1)}(\xi)$ (i.e. the solution of (4.7) when $\lambda = \lambda_1$) is the only eigenfunction which is positive on the whole interval $(0, 1)$. This therefore gives the solution of the physical problem. Since the kernel $K$ depends on $c$ and $\alpha$, we find $\lambda_1(c, \alpha)$ from (4.7) and must finally determine $c(\alpha, \gamma)$ numerically by inserting this value for $\lambda$ in (4.9). As a first step we replace the integral equation by a set of linear algebraic equations.

## Solution of the integral equation (4.7)

The separable form of the kernel suggests a solution of the form

$$\phi(\xi) = \sum_{m=0}^{\infty} \alpha_m \, c^m \, \phi_m(\xi) \; . \tag{4.13}$$

Then since the $\{\phi_m\}$ are linearly independent, the coefficients of $c^m \phi_m(\xi)$ on the two sides of the equation can be equated. This gives a set of linear equations for the $\{\alpha_m\}$ which can be written

$$\lambda \alpha_m = \sum_{n, p=0}^{\infty} c^{m+2n} \, h_{mn} \, i_{np} \, c^p \, \alpha_p \tag{4.14}$$

where

$$i_{np} = \frac{\sin \pi a}{\pi g(a)} \int_0^1 \eta^{a+n-1} \phi_p(\eta) d\eta \; .$$

Now a detailed calculation in appendix C shows that

$$i_{np} = h_{np} \; , \tag{4.15}$$

so (4.14) can be written in matrix form as

$$\lambda \underset{\sim}{a} = K^2 \underset{\sim}{a} \tag{4.16}$$

where $K = CHC$, $H = \{h_{mn}\}$, $C = \begin{pmatrix} 1 & & \\ & c & \\ & & c^2 \\ & & & \ddots \end{pmatrix}$ being infinite matrices.

Since $K$ and $K^2$ are positive, if $\lambda_1$ is the Perron root of $K^2$ it follows that $\sqrt{\lambda_1}$ is the Perron root of $K$, the corresponding eigenvector $\alpha^{(1)}$ and eigenfunction $\phi^{(1)}(\xi) = \sum \alpha_m^{(1)} \phi_m(\xi)$ being the same in each case. Moreover, $\phi^{(1)}(\xi)$ is the first eigenfunction of the simpler equation

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$$\sqrt{\lambda}\ \phi(\xi) = \int_0^1 \eta^{\alpha-1}\ \tilde{K}(\xi,\eta)\ \phi(\eta)d\eta \tag{4.17}$$

where $\quad \left(\dfrac{\pi g(\alpha)}{\sin \pi \alpha}\right)\tilde{K}(\xi,\eta) = L^{-1}(\xi)(1-c^2\xi\eta)^{-1} = \displaystyle\sum_{m=0}^{\infty} c^{2m}\ \phi_m(\xi)\eta^m\ .$

Because of (4.15), it follows from (4.8) that

$$\int_0^1 \tilde{K}(\xi,\zeta)\ \tilde{K}(\zeta,\eta)\ \zeta^{\alpha-1}d\zeta = K(\xi,\eta)\ . \tag{4.18}$$

[An equivalent result is $\displaystyle\int_0^1 \zeta^{\alpha-1}(1-c^2\xi\zeta)^{-1}\tilde{K}(\zeta,\eta)d\zeta = H(\xi,\eta).$ \qquad(4.19)]

So all eigenfunctions of (4.17) are also eigenfunctions of (4.7). The reverse does not follow in general although it is true for the Perron root. It may be noted that (4.17) can also be written in the form

$$\sqrt{\lambda}\ L\phi(\xi) = \frac{\sin \pi \alpha}{\pi g(\alpha)}\int_0^1 \frac{\eta^{\alpha-1}\ \phi(\eta)d\eta}{1-c^2\xi\eta} \tag{4.20}$$

in which the left hand side equals $\sqrt{\lambda}\ \psi(\xi)$, but the author has not been able to derive this equation directly from the physical problem.

## 5. Calculation of the Perron root $\lambda_1$ and the physical eigenvalue

The matrix $H$ formed by the coefficient $h_{mn}$ $(m, n = 0, 1, 2, \ldots)$ in equation (4.5) is the product

$$H = BA \tag{5.1}$$

where $\quad A = \begin{pmatrix} 1 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} \qquad B = \begin{pmatrix} 1 & & & \\ b_1 & 1 & & \\ b_2 & b_1 & 1 & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

are respectively an infinite Hankel matrix and an infinite lower triangle, the a's being the coefficients in the expansion of $A(x)$ already quoted (equation (3.6)) and the b's those in the expansion of $(1-x)^{\frac{1}{2}-\alpha}$, namely

$$b_k = (\alpha - \tfrac{1}{2})_k / k! \, . \qquad (5.2)$$

The fact that all elements of $H$ are positive is deduced as follows:

(i) The $m^{th}$ element in the left hand column is

$$h_{m0} = b_m + b_{m-1}a_1 + \dots a_m = \text{coefficient of } x^m \text{ in expansion of } (1-x)^{\frac{1}{2}-\alpha}A(x) \, .$$

This function equals $F(\tfrac{1}{2} + \alpha, \alpha; 1 + \alpha; x)$ by Kummer's transformation (Erdélyi et al 1953, p. 105 §2.9 (2)),

whence

$$h_{m0} = \frac{\alpha}{m+\alpha} \frac{(\tfrac{1}{2}+\alpha)m}{m!} > 0 \, , \qquad (5.3)$$

(ii) $h_{m-1, n+1} - h_{mn} = -b_m a_n > 0$ for all $m > 0$ (since all the b's except $b_0$ are negative) so the elements of $H$ increase to the right along lines of constant $m + n$, so are all positive.

The Perron root of $K$ is $\sqrt{\lambda_1}$, and may be computed by the power method of successive multiplication by $K$ until convergence in achieved. In numerical work, $K$ was first truncated to $K_N$ containing the first $N$ rows and columns, leading to a value $(\sqrt{\lambda_1})_N$; the procedure was then repeated for $N + 5$ until the value of $\sqrt{\lambda_1}$ did not change by more than $10^{-6}$. With increasing $c$, the value of $N$ to secure convergence increased from 10 at $c^2 = 0.1$ to 45 at $c^2 = 0.95$.

Finally, the solution of the physical problem in the form of $\alpha$ as a function of $c$ and $\gamma$ is found by iteration from (4.9), written logarithmically in the form

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$$\alpha = \frac{\frac{1}{2}\log(1-\gamma^2}{\log c + \dfrac{\log \lambda_1(c,\alpha)}{2\alpha} + \dfrac{1}{\alpha}\log g(\alpha)} . \qquad (5.4)$$

Starting from a trial value of $\alpha$ on the right, with $c$ and $\gamma$ held fixed, this converges very rapidly. Results are given in table 1.

The friction coefficient is then found as

$$\mu = (1/\gamma)\tan \pi\alpha .$$

Values of $\mu$ so found from the values of $\alpha$ given in table 1 are listed in table 2 for the cases $\nu = 0$ and $\nu = 0.3$, and plotted as solid lines in figure 2 .

### 5.1. Limiting form of solution as $\alpha \to 0$

In the limit $\alpha \to 0$, $\dfrac{\log \lambda_1}{2\alpha}$ tends to a limit $\lambda^*(c)$ say, as can be inferred directly from the matrix (see footnote page 21), or from the limiting solution of section 2.2.

The limit can be approached by allowing $\gamma$ to tend to zero, and since $(1/\alpha)\log g \to -\log 4$, (5.4) gives

$$\lim_{\gamma \to 0}\left[\frac{\tan^{-1}\mu\gamma}{\log(1-\gamma^2)}\right] = \frac{\pi}{2}\bigg/(\log c + \mu^* - \log 4) . \qquad (5.5)$$

The limit on the left is $\mu/\gamma = K(c)/K'(c)$ by (2.18), whence

$$\mu^*(c) = \log\frac{4}{c} - \frac{\pi}{2} K'(c)/K(c) . \qquad (5.6)$$

A convenient table is given on page 608 of Abramowitz and Segun.

Table 1 shows that $(\log \lambda_1)/\alpha$ differs appreciably from $\lambda^*$ only when $c$ is close to 1 and $\gamma$ close to $\frac{1}{2}$; for $\nu = 0.3$ it is sufficiently accurate to write $\lambda^*$ in place of $(\log \lambda_1)/\alpha$ so the determination of $\lambda_1$ from the matrix can be avoided, although iteration to find $\alpha$ is still necessary since $(\log g)/\alpha = -\log 4$ is not sufficiently accurate. The full expansion is

$$\log g(\alpha) = -\alpha \log 4 + \sum_{n=2}^{\infty} (-2\alpha)^n \frac{\eta(n)}{n} \qquad (5.7)$$

where $\eta(n)$ is the sum of reciprocal $n^{th}$ powers with alternating signs $(\eta(2) = \pi^2/12)$, and if one extra term is retained, (5.1) can then be written

$$\pi \alpha \Big/ \log(1 - \gamma^2)^{-1} = \left[ \frac{K'(c)}{K(c)} - \frac{\pi \alpha}{3} \right]^{-1} \qquad (5.8)$$

which leads to values of $\mu$ in 4-figure agreement with those tabulated in most cases. This expression was used to calculate the entries in table 2 for $c < 0.7$.

The solution for $\phi(\xi)$ can be written down explicitly when $\alpha \to 0$, since in this limit the first eigenvector of the matrix $K$ is $(1, 0, 0, \dots)$ (This is because in the limit all the elements of the left hand column of $K$ are zero, except for $k_{00}$).

Therefore, as $\alpha \to 0$

$$\phi^{(1)}(\xi) = \lim_{\alpha \to 0} \phi_0(\xi)$$

$$= 1 - \frac{2}{\pi} \xi^{\frac{1}{2}} F(\tfrac{1}{2}, \tfrac{1}{2}; {}^3/2; \xi), \text{ by (4.11)},$$

$$= 1 - \frac{2}{\pi} \sin^{-1} \xi^{\frac{1}{2}} \qquad (5.9)$$

in perfect agreement with (2.20) when allowance is made for the change of notation.

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## Footnote:

To infer the existence of the limit directly from the matrix, note that all the elements of the left hand column of K except the first contain $\alpha$ as a factor. Therefore if we write

$$\lambda_1 = 1 + 2\alpha \tilde{\lambda},$$

as $\alpha \to 0$, $\tilde{\lambda}$ tends to $\lambda^*(c)$, given by

$$\det \begin{pmatrix} -\lambda^* & c^2 h_{01}^* & c^4 h_{02}^* & \cdots \\ (h_{10}/\alpha)^* & c^2 h_{11}^* - 1 & c^4 h_{12}^* & \cdots \\ (h_{20}/\alpha)^* & c^2 h_{21}^* & c^4 h_{22}^* - 1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} = 0$$

where the stars denote the limits of the quantities as $\alpha \to 0$, so $(h_{m0}/\alpha)^\alpha = (\frac{1}{2})_m/m.$ m! etc.

I have not found a direct derivation of (5.6) from this determinant, but have satisfied myself that the two expressions for $\lambda^*$ are the same with 6 figure accuracy by means of a computation, which took only a fraction of a second of machine time.

## 5.2. Asymptotic expression as $c \to 1$

The writer has also carried out an analysis parallel to that reported here, in which the problem is cast into the form of an integral equation for the unknown strain $u'(x)$ in the slip region $c < x < 1$ . This is tractable by the Wiener-Hopf technique when $c$ is close to 1, and as in the present approach, an eigenvalue problem for $c$ follows from the requirement of bounded stresses at $x = c$, giving rise to the condition

$$\frac{\tan^{-1} \mu}{\kappa} = \tanh^{-1} c + \Psi(\alpha, \kappa) + \log 2 \tag{5.10}$$

with an error of order $(1 - c)^2$, where

$$\kappa = (2/\pi) \tanh^{-1} \gamma = (1/\pi) \ln(3 - 4\nu)$$

and

$$\Psi(\alpha, \kappa) = (1/\kappa) \operatorname{Im} \log[\Gamma(1 - \alpha - \frac{i\kappa}{2})/\Gamma(1 - \frac{i\kappa}{2})]$$

$$= \frac{\alpha \pi^2}{12}[1 + 0.2326 \pi \alpha + \frac{\pi^2}{15}(\alpha^2 - \frac{\kappa^2}{4}) + \dots] \ .$$

Values of $\mu$ calculated from this expression are included in table 2 and are seen to be extremely accurate for $c$ greater than about 0.6 .

## 5.3. Numerical solution of the coupled equations

Mention may also be made of a numerical attack on the problem, which will be described separately in an MRC Technical Summary Report by the author. If we write $y$ for $x$, and apply the operators $\int_0^x \frac{}{\sqrt{x^2 - y^2}}, \int_0^x \frac{}{\sqrt{x^2 - y^2}}$ to (2.11) and (2.12) respectively, they are put in the form of coupled Volterra equations:

$$0 < x < 1: \quad \int_x^1 \frac{t \, p_0(t) dt}{\sqrt{t^2 - x^2}} + \gamma \int_0^x \frac{x \, y \, q_0(y) dy}{\sqrt{x^2 - y^2}} = \frac{1}{2} \tag{5.11}$$

$$0 < x < c: \quad \gamma \int_0^x \frac{p_0(x)dy}{\sqrt{x^2 - y^2}} - \int_x^1 \frac{q_0(t)dt}{\sqrt{t^2 - x^2}} = 0 \qquad (5.12)$$

which are particularly suitable for numerical solution, as the earlier singular equations are not. They have been solved numerically on the Univac 1108 computer at the University of Wisconsin Computing Center, using piecewise constant interpolations to p, q with up to 64 sub-intervals, the condition $q = \mu p$ being applied on the portion $c < x < 1$ with an initially guessed value of $\mu$. For each fixed c, an iteration was performed until the value of $\mu$ was such that $q - \mu p$ was positive throughout $(0, c)$ and tended to zero at c. In this way the points shown by circles in figure 2, and the corresponding stress distributions in figures 3 and 4 were obtained.

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## Appendix A. Inversion of singular integral equations

Carlemann (1922) showed that the general solution of the equation

$$a(s) u(s) - \lambda \int_{-1}^{1} \frac{u(t)dt}{t - s} = g(t) \qquad (A1)$$

$(\lambda > 0)$ where $a(s)$ and $g(s)$ are analytic for $-1 \le s \le 1$ can be written

$$u(s) = \frac{a(s)g(s)}{b^2(s)} + \frac{\lambda e^{\tau(s)}}{b(s)} \int_{-1}^{1} \frac{e^{-\tau(t)} g(t)dt}{b(t)(t-s)} + \frac{Ke^{\tau(s)}}{1-s}, \qquad (A2)$$

where $K$ is an arbitrary constant, and

$$\tau(s) = \int_{-1}^{1} \frac{\theta(t)dt}{t-s}, \quad \theta(s) = \frac{1}{2\pi i} \log\left[\frac{a(s) + \lambda\pi i}{a(s) - \lambda\pi i}\right] \quad (0 < \theta(s) < 1),$$

$$b^2(s) = a^2(s) + (\lambda\pi)^2.$$

Equations (2.16), (3.1) and (4.2) are all of this form, and the inverses quoted are obtained by suitable changes of variable and choice of $K$: eg. in (3.1) write $x^2 = \frac{1}{2}(s+1)$, $y^2 = \frac{1}{2}(t+1)$, $x^{-1}p(x) = u(s)$, $x^{-1}r(x) = -g(s)$, when the equation becomes identical with (A1) with $a = -1$, $\pi\lambda = \cot \pi\alpha$. Then $\theta(s) = \frac{1}{2} + \alpha$, $\exp\tau(s) = \left(\frac{1-s}{1+s}\right)^{\frac{1}{2}+\alpha}$, $b(s) = \operatorname{cosec} \pi\alpha$ and the solution (A2), translated back into the original variables, is

$$(\operatorname{cosec} \pi\alpha)p(x) = (\sin \pi\alpha)r(x) - 2\left(\frac{\cos \pi\alpha}{\pi}\right) \int_{0}^{1} \left(\frac{1-x^2}{1-y^2}\right)^{\frac{1}{2}+\alpha} \left(\frac{y}{x}\right)^{2\alpha} \frac{yr(y)dy}{y^2 - x^2}$$

$$+ \tfrac{1}{2}Kx^{-2\alpha}(1-x^2)^{-\frac{1}{2}+\alpha}.$$

If we choose

$$K = \frac{4 \cos \pi a}{\pi} \int_0^1 \frac{y^{2a-1} r(y) dy}{(1-y)^{1+a}} ,$$

the last two terms combine to give the expression for $p(x)$ quoted in (3.3),

which is regular at $x = 0$.

## Appendix B. Evaluation of the integrals (3.5) and (4.11)

These are special cases of the following general result for real $a, b < 1$, $0 < s < 1$:

$$\frac{\cos \pi a}{s^a (1-s)^b} - \frac{\sin \pi a}{\pi} \int_0^1 \frac{dt}{t^a (1-t)^b (t-s)} = \frac{\Gamma(1-b)}{\Gamma(1+a)\Gamma(1-a-b)} F(1, a+b; a+1; s) . \quad (B1)$$

This is obtained by evaluating the loop integral $\frac{1}{2\pi i} \int_{+\infty}^{(0-)} dz/z^a (z-1)^b (z-s)$ round the two sides of the positive real axis in the $z$ plane, with semi-circular indents above and below the point $z = s$. The second term on the left hand side is the contribution from the two sides of the interval $(0, 1)$, and the first that from the indents at $z = s$. The right hand side comes from the integral on $(1, \infty)$, which can be transformed into Euler's integral for the hypergeometric function. (In the special case $a + b = 0$, the contour can be replaced by a closed loop surrounding the points $0, 1$, and the right hand side, which reduces to 1, is the residue at infinity.) From this result

(i) (3.5) is obtained immediately by writing $a = \alpha$, $b = \frac{1}{2} - \alpha$, $s = x^2$, $t = y^2$,

(ii) To derive (4.11), write $\psi = 1$ in (4.6) whence

$$\phi_0(\xi) = \frac{\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}-\alpha}}{\sin \pi \alpha} \left[ \frac{\sin \pi \alpha}{\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}-\alpha}} + \frac{\cos \pi \alpha}{\pi} \int_0^1 \frac{d\eta}{\eta^{\frac{1}{2}}(1-\eta)^{\frac{1}{2}-\alpha}(\eta-\xi)} \right] .$$

Use of (B1) with $a = \frac{1}{2} - \alpha$, $b = \frac{1}{2}$, $s = 1 - \xi$, $t = 1 - \eta$ shows that the quantity in square brackets is

$$[\Gamma(\tfrac{1}{2})/\Gamma(\alpha)\Gamma(\tfrac{3}{2} - \alpha)] F(1, 1-\alpha, \tfrac{3}{2} - \alpha; 1-\xi) ,$$

from which the first line of (4.11) follows since $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin \pi \alpha$.

The second line is obtained from the first by writing $a = 0$, $b = \alpha$, $c = \frac{1}{2}$ in lines (22), (43), (1) and (17) of Erdélyi §2.9, pp. 105-108.

**Appendix C.** Evaluation of $i_{pq} = \dfrac{\sin \pi \alpha}{\pi g(\alpha)} \displaystyle\int_0^1 \xi^{\alpha+p-1} \phi_q(\xi) d\xi$

Substitution from (4.10) and the first line of (4.11) gives

$$i_{pq} = \frac{\alpha}{\Gamma(\tfrac{1}{2}+\alpha)\Gamma(\tfrac{3}{2}-\alpha)} \int_0^1 \xi^{p+q+\alpha-\tfrac{1}{2}}(1-\xi)^{\tfrac{1}{2}-\alpha} \sum_{n=0}^{\infty} \frac{(1-\alpha)n}{(\tfrac{3}{2}-\alpha)_n}(1-\xi)^n d\xi \qquad (1)$$

$$+ \frac{\cos \pi\alpha}{\pi} \int_0^1 \xi^{\tfrac{1}{2}}(1-\xi)^{\tfrac{1}{2}-\alpha} \sum_{r=0}^{q-1} a_{q-r-1} \xi^{r+\alpha+p-1} d\xi . \qquad (2)$$

On termwise integration,

$$(1) = \frac{\alpha\Gamma(p+q+\alpha+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}+\alpha)\,\Gamma(p+q+2)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n}{(p+q+2)_n} .$$

The sum of the series is $F(1, 1-\alpha; p+q+2; 1)$, which equals $(p+q+1)/(p+q+\alpha)$ by Erdélyi 1953, p. 61(14), so

$$(1) = \left(\frac{\alpha}{p+q+\alpha}\right) \frac{(\alpha+\tfrac{1}{2})p+q}{(p+q)!} = \sum_{s=0}^{p+q} a_s b_{p+q-s} \qquad \text{by } (5.3) .$$

Similarly, integration of the second term gives

$$(2) = \frac{\cos\pi\alpha}{\pi} \sum_{r=0}^{q-1} a_{q-r-1}\left(\frac{-\pi}{\cos\pi\alpha}\right) b_{p+r+1} = -\sum_{s=0}^{q-1} a_s b_{p+q-s} .$$

Adding the two expressions gives

$$i_{pq} = \sum_{s=q}^{p+q} a_s b_{p+q-s} = h_{pq} , \qquad \text{by } (5.1) .$$

An alternative demonstration of this result is possible by direct integration of various hypergeometric functions to obtain the equivalent equation (4.19), but the analysis is more lengthy.

## Appendix D. Elliptic integrals for limiting stress distribution

If the order of integration is reversed in the first term on the right of (2.17),
we have

$$q_0(x) = \frac{2x}{\pi^2 \sqrt{c^2 - x^2}} \left[ \mu \int_c^1 \frac{t\, dt}{\sqrt{1-t^2}} I_0(x,t) - \gamma \int_0^c \sqrt{\frac{c^2-y^2}{1-y^2}} \frac{dy}{y^2-x^2} \right],$$

where $I_0(x,t) = \frac{2}{\pi} \int_0^c \frac{(c^2-y^2)^{\frac{1}{2}} dy}{(t^2-y^2)(y^2-x^2)} = \frac{-t^{-1}(t^2-c^2)^{\frac{1}{2}}}{t^2-x^2}$ when $0 < x < c < t < 1$

so $$q_0(x) = \frac{2x}{\pi^2 \sqrt{c^2-x^2}} \left[ -\mu \int_c^1 \left( \frac{t^2-c^2}{1-t^2} \right)^{\frac{1}{2}} \frac{dt}{t^2-x^2} - \gamma \int_0^c \left( \frac{c^2-y^2}{1-y^2} \right)^{\frac{1}{2}} \frac{dy}{y^2-x^2} \right],$$

exhibiting a singularity at $x = c$ unless the sum inside the square bracket
vanishes at this point, i.e. unless

$$0 = -\mu \int_c^1 \frac{dt}{(1-t^2)^{\frac{1}{2}} (t^2-c^2)^{\frac{1}{2}}} + \gamma \int_0^c \frac{dy}{(1-y^2)^{\frac{1}{2}} (c^2-y^2)^{\frac{1}{2}}}$$

which gives $\mu/\gamma = K(c)/K'(c)$, equation (2.18). When this is satisfied, the
previous expression can be rearranged as

$$q_0(x) = \left( \frac{2x}{\pi^2} \right) \sqrt{c^2-x^2} \left[ \mu I_1(x) - \gamma I_2(x) \right],$$

where $I_1(x) = \int_c^1 \frac{dt}{(1-t^2)^{\frac{1}{2}} (t^2-c^2)^{\frac{1}{2}} (t^2-x^2)} = \frac{1}{1-x^2} \Pi_1\left( \frac{1-c^2}{1-x^2}, \sqrt{1-c^2} \right)$

$$= \frac{1}{x\sqrt{(c^2-x^2)(1-x^2)}} \left[ E'F(\xi,c) + K'E(\xi,c) - K'F(\xi,c) \right]$$

and $I_2(x) = \int_0^c \dfrac{dy}{\sqrt{(1-y^2)(x^2-y^2)(y^2-x^2)}} = -\dfrac{1}{x^2}\, \Pi_1\left(\dfrac{c^2}{x^2}, c\right)$

$$= \dfrac{1}{x\sqrt{(c^2-x^2)(1-x^2)}}\, \left[KE(\xi,c) - EF(\xi,c)\right] \ .$$

Here $\Pi_1$ is the complete elliptic integral of the third kind (Byrd and Friedmann, p. 225), $\xi = \sin^{-1}\dfrac{x}{c}$, $K = K(c)$ and $E = E(c)$ are the complete integrals of the first and second kinds, and dashes denote the same integrals with argument $\sqrt{1-c^2}$ . $E(\xi,c)$, $F(\xi,c)$ are the incomplete elliptic integrals $\int_0^\xi (1-c^2 \sin^2\theta)^{\pm\frac{1}{2}}\, d\theta$ . Then since $\gamma = \mu K'/K$, these results combine to give

$$q_0(x) = \dfrac{2\mu}{\pi^2\sqrt{1-x^2}}\, [KE' + K'E - KK']\dfrac{F(\xi,c)}{K(c)} \ .$$

The quantity in the square brackets equals $\pi/2$ by Legendre's formula, so $q_0(x)$ reduces to the expression quoted (2.19).

# TABLE 1

Iteration to find $a$, for $\nu = 0$ $(\gamma = \frac{1}{2})$ and $\nu = 0.3(\gamma = \frac{2}{7})$

| $c^2$ | N | $\lambda^*(c)$ (equation 5.5) | $\nu = 0$ | | $\nu = 0.3$ | |
|-------|---|------|-----------------------|-----------|-----------------------|-----------|
| | | | $\dfrac{\log \lambda_1}{2a}$ | $a$ | $\dfrac{\log \lambda_1}{2a}$ | $a$ |
| .1 | 10 | .026080 | .025997 | .059399 | .026069 | .017140 |
| .2 | 10 | .054620 | .054370 | .070821 | .054586 | .020233 |
| .3 | 10 | .086190 | .085680 | .080634 | .086155 | .022824 |
| .4 | 15 | .121604 | .120699 | .090332 | .121521 | .025303 |
| .5 | 20 | .162072 | .160555 | .100748 | .161932 | .027886 |
| .6 | 25 | .209533 | .207058 | .112806 | .209308 | .030765 |
| .7 | 30 | .267427 | .263323 | .128027 | .267064 | .034229 |
| .75 | 30 | .302242 | .296837 | .137751 | .301768 | .036342 |
| .8 | 35 | .342929 | .335596 | .149923 | .342307 | .038884 |
| .85 | 35 | .392360 | .381957 | .166332 | .391447 | .042121 |
| .9 | 40 | .456536 | .440404 | .191387 | .455181 | .046695 |
| .95 | 45 | .552638 | .521275 | .241681 | .549429 | .054643 |

## TABLE 2

### Friction coefficient μ as a function of c

| c | $\nu = 0$ | | $\nu = 0.3$ | |
|---|---|---|---|---|
| | μ (equation 5.4) | μ (equation 5.10) | μ (equation 5.4) | μ (equation 5.10) |
| .05 | .20952 | | .10728 | |
| .1 | .25064 | | .12771 | |
| .15 | .28356 | | .14384 | |
| .2 | .31313 | | .15815 | |
| .25 | .34118 | | .17154 | |
| .3 | .36869 | | .18449 | |
| .35 | .39633 | | .19733 | |
| .4 | .42466 | | .21028 | |
| .45 | .45422 | | .22356 | |
| .5 | .48557 | | .23738 | |
| .55 | .51941 | | .25199 | |
| .6 | .55660 | | .26766 | |
| .6325 | .58332 | | .27882 | |
| .7071 | .65504 | .65722 | .30741 | .30819 |
| .7746 | .74003 | .74142 | .33934 | .33973 |
| .8367 | .85080 | .85168 | .37782 | .37805 |
| .8660 | .92393 | .92465 | .40135 | .40150 |
| .8944 | 1.01845 | 1.01914 | .42969 | .42978 |
| .9220 | 1.15190 | 1.15260 | .46587 | .46597 |
| .9487 | 1.37198 | 1.37300 | .51715 | .51720 |
| .9747 | 1.89811 | 1.90518 | .60680 | .60756 |
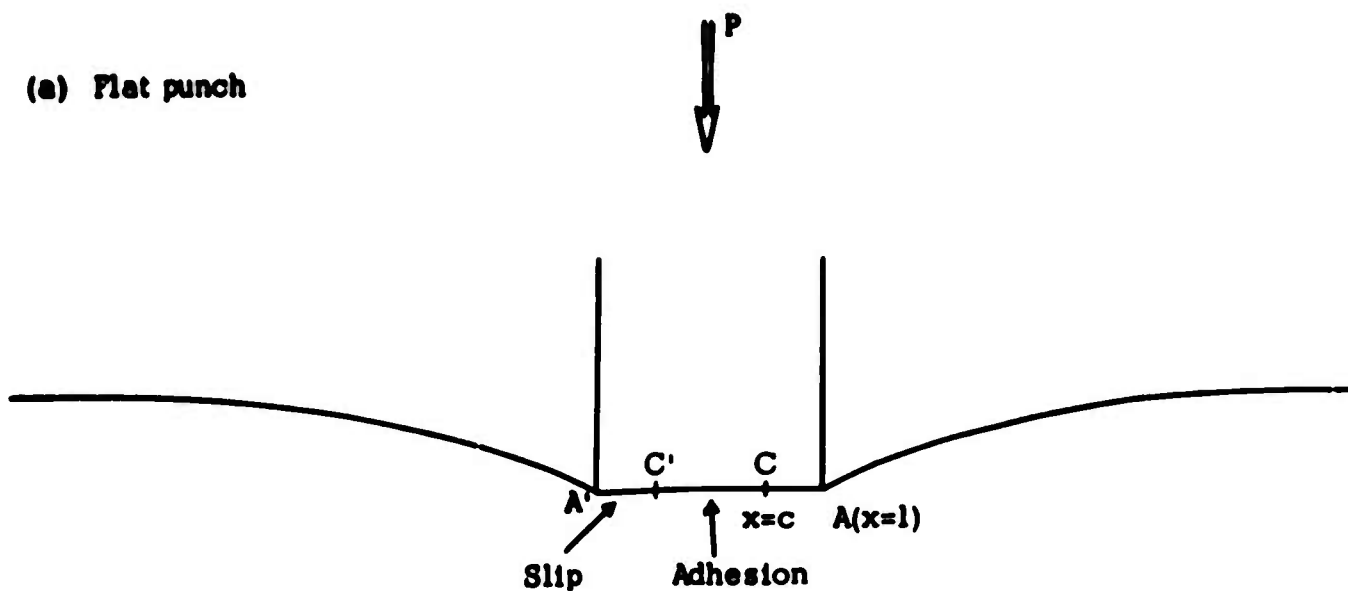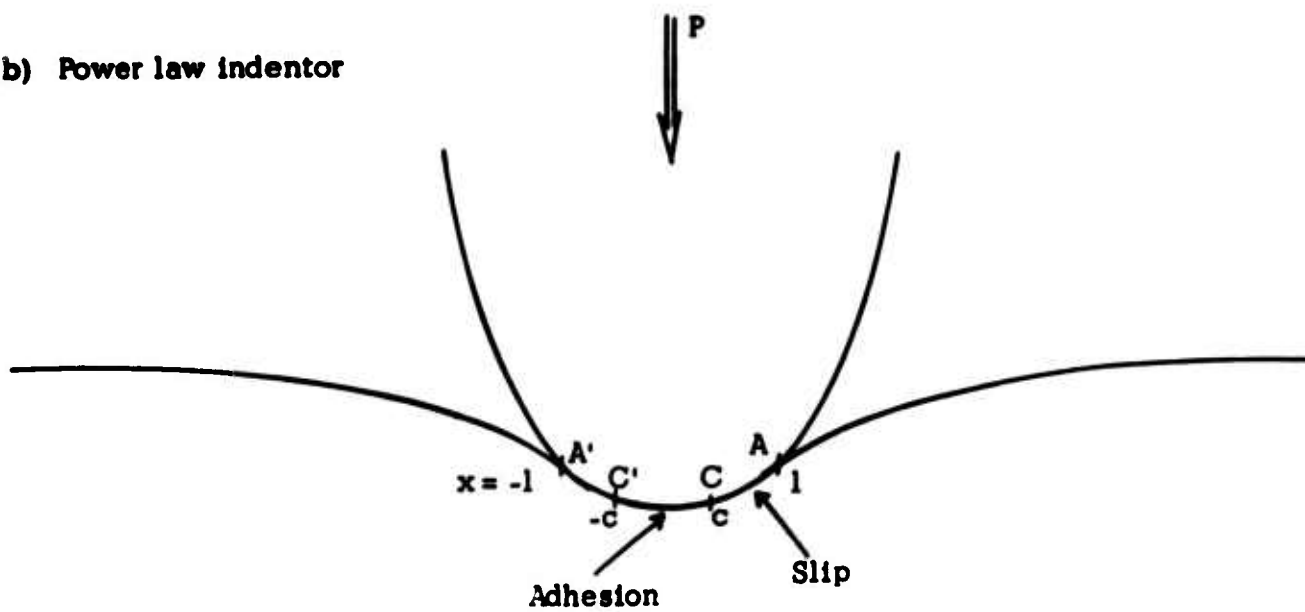
Figure 1: Frictional indentation by rigid punch
(Schematic)

Figure 2: Two dimensional punch: μ versus c
for $\nu = 0$ and $\nu = 0.3$

——————— Exact

– – – – – Galin

• • • • • • • • • approximate (equation 2.18)

⊙ Computed using Abel equations (5.11, 5.12)
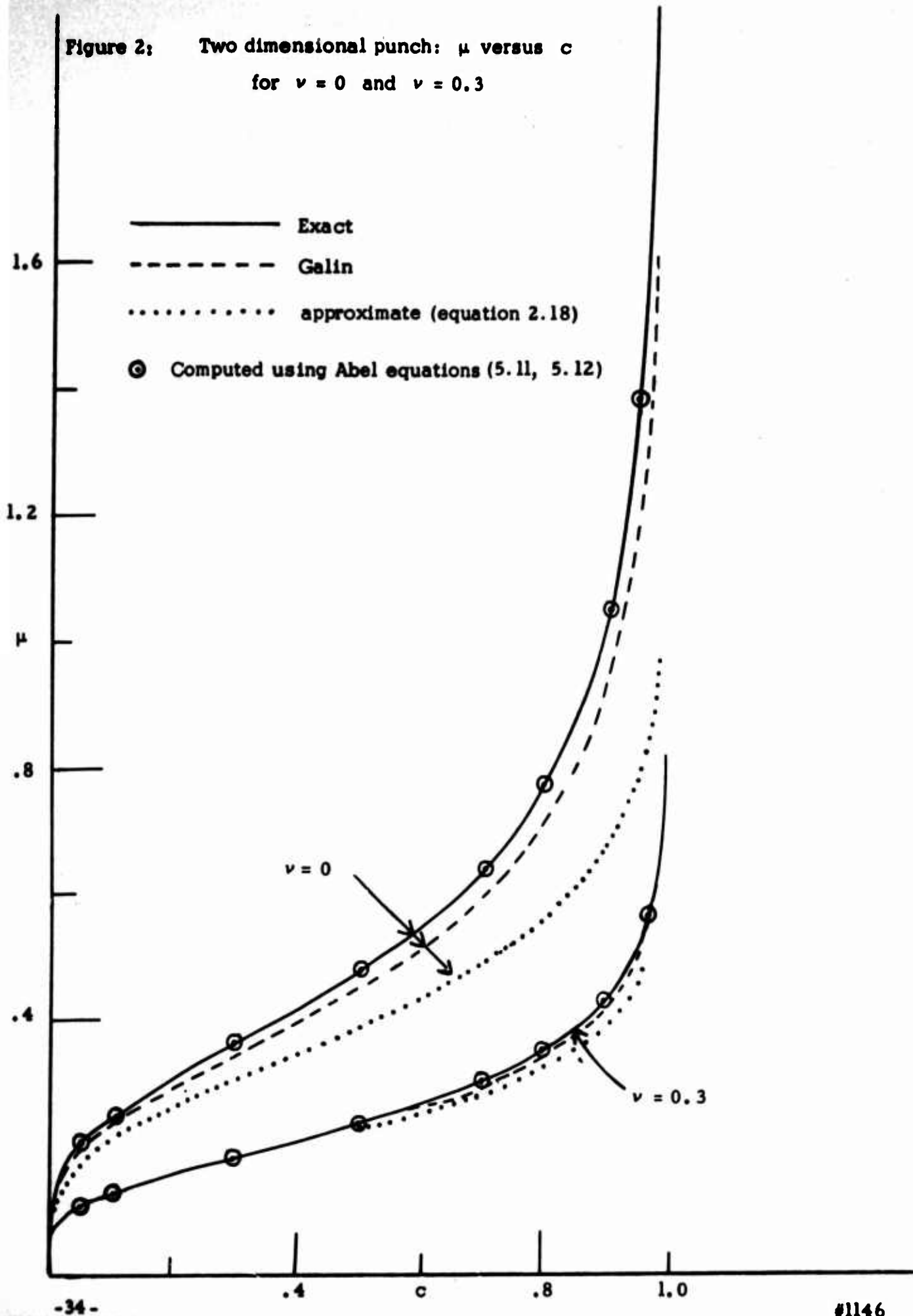
$\nu = 0$

$\nu = 0.3$

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Figure 3: Shear stress distribution for flat indentor, $\nu = 0$ (solid lines) Broken lines are curves of $\frac{2}{\pi} \sin^{-1} \frac{x}{c}$ (equation 2.21)
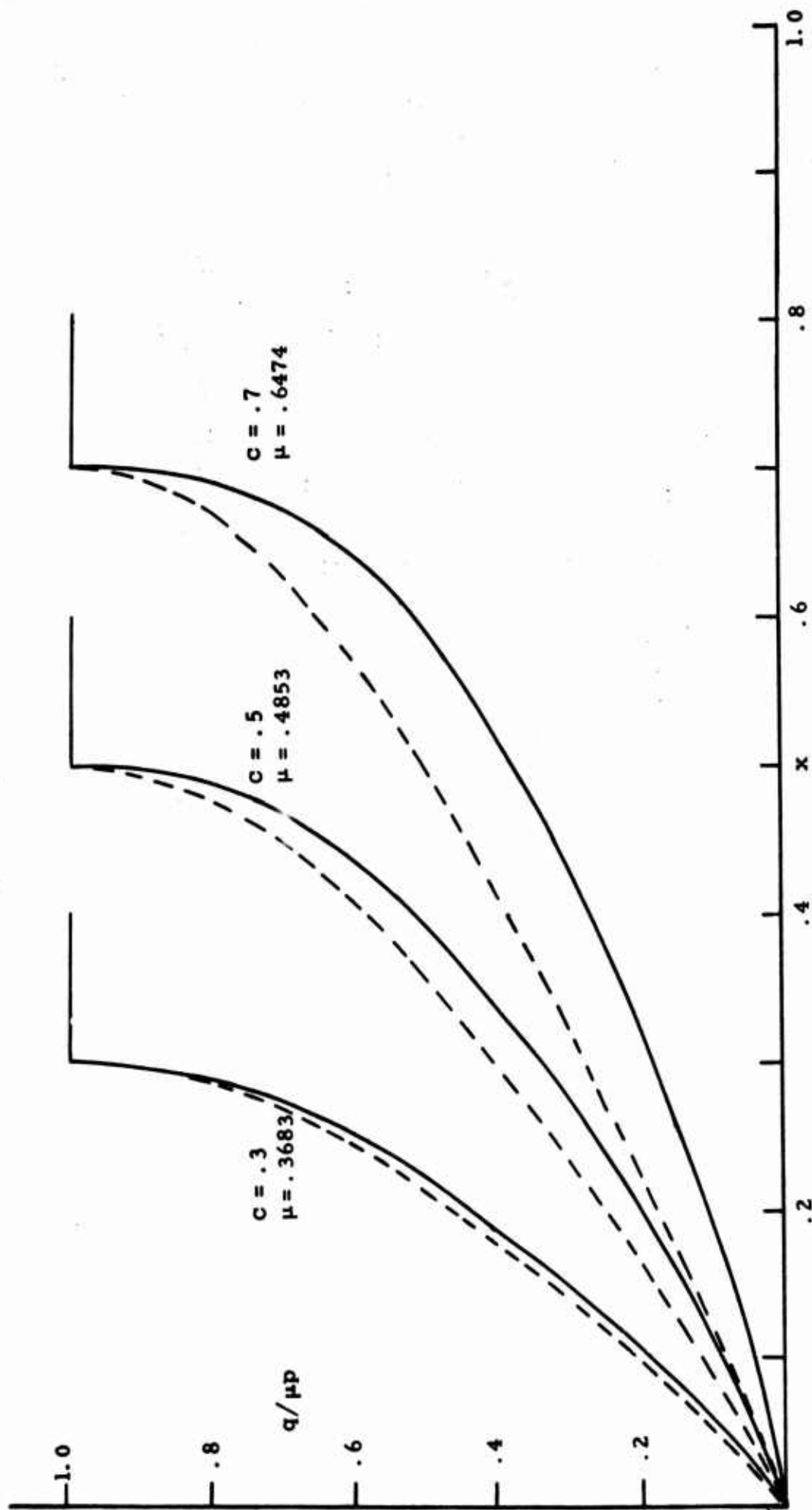
#1146

Figure 4:

Normal stress distributions for $\nu = 0$: two-dimensional flat punch

fully adhesive

$\mu = .647, c = .7$

.368, .3

.208, .05

Frictionless

Limiting friction ($\tau = \mu\sigma$)

p

x

.3  .2  .4  .6  .8  1.0

.3  .5  .6

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